The Riemann Zeta Function

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1 Introduction

The Riemann Zeta Function $\zeta(s)$ is defined on the half-plane $\{\Re(s) > 0\}$ as the convergent sum $\sum_{n=1}^{\infty} \frac{1}{n^s}$.

Exercise. Prove the product

$$\prod_p \frac{1}{1 - \frac{1}{p^s}}$$

converges in this half-plane, and equals $\zeta(s)$ there.

There is an analytic continuation of this function, discussed in the next section, which allows us to extend $\zeta(s)$ to a meromorphic function on $\mathbb{C}$ with a simple pole at $s = 1$ with residue 1.

Here are some values of $\zeta(s)$:

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \text{and in general, } \zeta(2k) = |B_{2k}| 2^{2k-1} \pi^{2k}/(2k)!$$

(See the next session for the definition of the B_n's.)

$$\zeta(0) = -\frac{1}{2},\ \zeta(-1) = -\frac{1}{12},\ \zeta(-3) = \frac{1}{120}.$$

$$\text{In general, } \zeta(n) = \begin{cases} 0 \text{ if } n \text{ is odd} \\ -\frac{B_{2k}}{2k} \text{ if } n = -2k+1 \end{cases}$$

2 Analytic Continuation

2.1 The Euler-MacLaurin Summation Formula

If f is a polynomial function, then

$$\sum_{n=1}^{N} f(n) = \int_{1}^{N+1} f(x)dx + \sum_{n=1}^{\infty} \frac{B_n}{n!}\{f^{(n-1)}(N+1) - f^{(n-1)}(1)\}.$$

$$\text{Here, } \sum_{n=0}^{\infty} \frac{B_n z^n}{n!} = \frac{z}{e^z - 1}.$$

$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}$, and other Bernoulli numbers are 0.

A recursive formula for computing the Bernoulli numbers is

$$\sum_{n=0}^{N} \binom{N+1}{i} B_i = 0.$$

This gives the familiar formulas for sums of the form $\sum_{n=1}^{N} n^k$.

Exercise. Show that this formula does not hold for $f(x) = \frac{1}{x}$.

To extend this theory to arbitrary smooth functions, one must use the Bernoulli polynomials.

We define $B_n(x)$ so that

$$\sum_{n=0}^{\infty} \frac{B_n(x) z^n}{n!} = \frac{ze^x z}{e^z - 1}$$

$$B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6}, B_3(x) = x^3 - \frac{3x^2}{2} + \frac{x}{2}.$$

A explicit formula for computing the Bernoulli polynomials is

$$B_n(x) = \sum_{k=0}^{N} \binom{N}{k} B_{n-k} x^k.$$

Let $\overline{B}_n(x)$ denote the function of period one which agrees with $B_n(x)$ on [0,1]. Then the Euler-MacLaurin summation formula is

$$\sum_{n=1}^{N} f(n) = \int_1^{N+1} f(x)dx + \sum_{n=1}^{N} \frac{B_n}{n!}\{f^{(n-1)}(N+1) - f^{(n-1)}(1)\} + S_M(x,N),$$

$$\text{where } S_M(x,N) = \frac{(-1)^{M+1}}{M!}\int_1^{N+1} \overline{B}_M(x) f^{(M)}(x)dx.$$

Spivak's Calculus shows how this gives Stirling's formula.

Applying this to formula to $f(x) = \frac{1}{x^s}$ and letting N go to infinity, we obtain

$$\zeta(s) = \frac{1}{s-1}+\frac{1}{2}+\sum_{n=2}^{N}\frac{B_k s(s+1)...(s+n-2)}{n!} - /frac1M!s(s+1)...(s+M-1)\int_1^{\infty}\overline{B}_n(x)x^{-s-M}dx.$$

This allows us to extend $\zeta(s)$ to

$$\{\Re(s) > 1-M\}$$

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Exercise. Use this to compute the nonpositive values of $\zeta(s)$ from the previous section.

2.2 The Functional Equation

Riemann himself found two methods of proving the functional equation for $\zeta(s)$. One involves "Hankel contours" and one involves the Poisson Summation formula.

Either way you get

$$2^{1-s}\Gamma(1-s)\zeta(s)\cos(\frac{\pi s}{2} = \pi^s\zeta(1-s).$$

Tate's thesis has a nice way to explain these factors and generalize this.